

Baire's Category Theorem and the Banach–Steinhaus Theorem

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Abstract

In this paper I prove Baire's category theorem for complete normed vector spaces and use it to prove two important results in functional and real analysis: the existence of continuous functions that are differentiable nowhere, and the Banach–Steinhaus theorem.

1 Introduction

Definition 1.0. A subset M of a normed vector space X is *dense* if for every $x \in X$ there exists a sequence $(x_i)_{i \geq 1} \subseteq M$ such that $\lim_{i \rightarrow \infty} x_i = x$.

Definition 1.1. A subset N of a normed vector space X is *nowhere dense* if there is no nonempty open $U \subseteq X$ in which N is dense.

The whole proof of Baire's theorem runs on a single consequence of Definition 1.1, so it is worth isolating. For closed sets it is an equivalence, and both directions are used later.

Lemma 1.2. (a) A nowhere dense set contains no open ball. (b) Conversely, a closed set containing no open ball is nowhere dense.

Proof. (a) Let $N \subseteq X$ be nowhere dense and suppose, for contradiction, that $B(x, \varepsilon) \subseteq N$ for some $x \in X$ and $\varepsilon > 0$. Every point of $B(x, \varepsilon)$ then lies in N , so N is dense in the nonempty open set $B(x, \varepsilon)$, each point being its own constant sequence. This contradicts N being nowhere dense.

(b) Let N be closed and suppose it is dense in some nonempty open U . Then $U \subseteq \text{cl}(N) = N$, and U , being open and nonempty, contains an open ball; hence so does N . Contrapositively, a closed set containing no open ball is nowhere dense. $\square$

2 Baire's Category Theorem

Definition 2.0. A normed space X is a *Baire space* if no countable union of closed nowhere dense subsets of X contains an open ball.

Theorem 2.1 (Baire). Every complete normed space is a Baire space.

Proof. Let $(X, \|\cdot\|)$ be a complete normed space. For the sake of contradiction, assume there is a sequence of closed nowhere dense sets $(C_i)_{i \geq 1} \subseteq X$ such that $Y = \bigcup_{i \geq 1} C_i$ contains an open ball, say $B(x_1, \varepsilon_1) \subseteq Y$.

The plan is to burrow: to build a nested sequence of balls, the $(n+1)$ st of which avoids C_n entirely, and then to catch the point they close down on. Concretely, I claim there exist points x_n and radii $\varepsilon_n > 0$ such that for every $n \geq 1$,

$$B(x_{n+1}, \varepsilon_{n+1}) \subseteq B(x_n, \frac{\varepsilon_n}{2}) \setminus C_n, \quad \varepsilon_{n+1} \leq \frac{\varepsilon_n}{2}. \quad (1)$$

Base case. $B(x_1, \varepsilon_1/2)$ is an open ball, so by Lemma 1.2(a) it is not contained in the nowhere dense set C_1 . Choose $x_2 \in B(x_1, \varepsilon_1/2) \setminus C_1$. Since C_1 is closed, $X \setminus C_1$ is open, and its intersection with the open ball $B(x_1, \varepsilon_1/2)$ is open and nonempty — nonempty because it contains x_2 . Any point of an open set is the centre of a ball fully contained in that set, so there exists $\varepsilon_2 > 0$ with $B(x_2, \varepsilon_2) \subseteq B(x_1, \varepsilon_1/2) \setminus C_1$, and we may shrink ε_2 so that $\varepsilon_2 \leq \varepsilon_1/2$.

Inductive step. Given x_n and $\varepsilon_n > 0$, the same three moves apply verbatim with n in place of 1: the open ball $B(x_n, \varepsilon_n/2)$ is not contained in the nowhere dense C_n , so pick $x_{n+1} \in B(x_n, \varepsilon_n/2) \setminus C_n$;

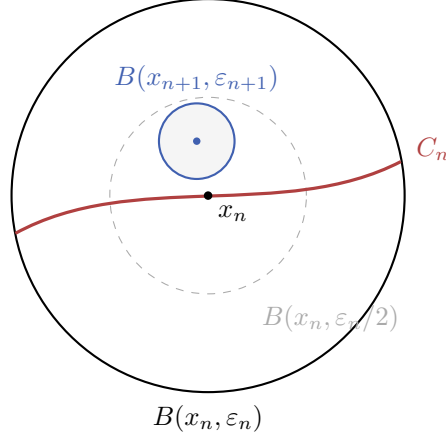


Figure 1: One step of the construction. The next ball is chosen inside the dashed half-ball and off to the side of C_n . Halving the radius is what keeps the limit point strictly inside every ball of the sequence.

that set is open because C_n is closed; so some $B(x_{n+1}, \varepsilon_{n+1})$ with $\varepsilon_{n+1} \leq \varepsilon_n/2$ sits inside it. This proves (1).

Two things follow from (1). First, the radii collapse: $\varepsilon_{n+1} \leq \varepsilon_n/2$ gives $\varepsilon_n \leq \varepsilon_1 2^{-(n-1)} \rightarrow 0$. Second, the balls nest, since $B(x_{n+1}, \varepsilon_{n+1}) \subseteq B(x_n, \varepsilon_n/2) \subseteq B(x_n, \varepsilon_n)$.

Hence $(x_n)_{n \geq 1}$ is Cauchy. Given $\varepsilon > 0$, pick k with $\varepsilon_k < \varepsilon$, which is possible since $\varepsilon_n \rightarrow 0$. For every $n \geq k$ nesting places x_n inside $B(x_k, \varepsilon_k)$, so $\|x_n - x_k\| < \varepsilon_k < \varepsilon$. Since $(X, \|\cdot\|)$ is complete, $x_n \rightarrow x$ for some $x \in X$.

Now locate the limit. Fix $n \geq 1$. For every $m > n$ we have $x_m \in B(x_{n+1}, \varepsilon_{n+1}) \subseteq B(x_n, \varepsilon_n/2)$, so $\|x_m - x_n\| < \varepsilon_n/2$, and letting $m \rightarrow \infty$ gives $\|x - x_n\| \leq \varepsilon_n/2 < \varepsilon_n$. This is exactly what the halved radius bought: passing to the limit costs a non-strict inequality, and the factor of two absorbs it, so x lands strictly inside $B(x_n, \varepsilon_n)$ rather than merely in its closure.

The same estimate at $n + 1$ gives $x \in B(x_{n+1}, \varepsilon_{n+1}) \subseteq X \setminus C_n$ for every $n \geq 1$, so x belongs to no C_n and therefore $x \notin Y$. But taking $n = 1$ gives $x \in B(x_1, \varepsilon_1) \subseteq Y$. Contradiction. $\square$

Corollary 2.2. A complete normed vector space is not a countable union of closed nowhere dense subsets.

Proof. A normed vector space contains at least the zero vector, so $B(0, 1)$ is an open ball contained in X . If X were a countable union of closed nowhere dense subsets, that union would contain an open ball, contradicting Theorem 2.1. $\square$

Completeness is doing real work here, and the statement is easy to get wrong without it. A general normed space really can be a countable union of closed nowhere dense pieces: take an infinite-dimensional space with a countable Hamel basis and write it as the union of the finite-dimensional subspaces spanned by the first n basis vectors, each of which is closed with empty interior.

Lemma 2.3. A subset M of a normed vector space X is dense if and only if $X \setminus M$ contains no open ball.

Proof. ($\Rightarrow$) Let M be dense and suppose $B(m, \varepsilon) \subseteq X \setminus M$ for some $m \in X$ and $\varepsilon > 0$. By density there is a sequence $(m_i)_{i \geq 1} \subseteq M$ with $m_i \rightarrow m$, so there is $N \geq 1$ with $\|m - m_N\| < \varepsilon$. Then $m_N \in B(m, \varepsilon) \subseteq X \setminus M$ while $m_N \in M$. These sets are disjoint, a contradiction.

($\Leftarrow$) Suppose $X \setminus M$ contains no open ball and let $x \in X$. For each $n \geq 1$ the set $B(x, 1/n)$ is an open ball, hence not contained in $X \setminus M$, so we may choose $m_n \in B(x, 1/n) \cap M$. Then $\|x - m_n\| < 1/n \rightarrow 0$, so $(m_n)_{n \geq 1} \subseteq M$ converges to x . As x was arbitrary, M is dense. $\square$

Lemma 2.3 is the translation dictionary between the two standard ways of stating Baire's theorem. Reading Theorem 2.1 through it gives the formulation that is most often quoted.

Lemma 2.4 (Alternate formulation of Baire's category theorem). In a complete normed space X , the intersection $\bigcap_{n \geq 1} O_n$ of any countable collection of open dense subsets $(O_n)_{n \geq 1}$ is dense in X .

Proof. Write $U = \bigcap_{n \geq 1} O_n$. Taking complements and applying De Morgan's laws,

$$X \setminus U = \bigcup_{n \geq 1} (X \setminus O_n).$$

Each O_n is open, so each $X \setminus O_n$ is closed; each O_n is dense, so by Lemma 2.3 each $X \setminus O_n$ contains no open ball; and a closed set containing no open ball is nowhere dense by Lemma 1.2(b). So $X \setminus U$ is a countable union of closed nowhere dense sets, and Theorem 2.1 says it contains no open ball. Applying Lemma 2.3 once more, this time to U , gives that U is dense. $\square$

We can now use Baire's theorem to prove a very interesting result in real analysis.

Theorem 2.5. There exists a continuous $f : [0, 1] \rightarrow \mathbb{R}$ that is differentiable at no point of $[0, 1]$. In fact the set of such functions is dense in $C([0, 1])$.

Proof. Consider $C([0, 1])$ equipped with the supremum norm $\|f\|_\infty = \sup_{x \in [0, 1]} |f(x)|$, under which it is complete. For f to be differentiable somewhere there must exist some $x \in [0, 1]$ at which the limit

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

exists. Inspired by this, define for each $n, m \geq 1$

$$A_{n,m} = \left\{ f \in C([0, 1]) : \exists x \in [0, 1] \text{ with } \left| \frac{f(x+h) - f(x)}{h} \right| \leq n \text{ for all } 0 < |h| < \frac{1}{m} \text{ with } x+h \in [0, 1] \right\}.$$

Every somewhere-differentiable function lands in one of these. Suppose f has derivative y at x , and choose $n > |y|$. By the definition of the limit there is $\varepsilon > 0$ such that $|(f(x+h) - f(x))/h| \leq n$ whenever $0 < |h| < \varepsilon$; now choose m with $1/m < \varepsilon$, which is clearly possible since $1/m \rightarrow 0$ as $m \rightarrow \infty$. Then $f \in A_{n,m}$. Writing $ND([0, 1])$ for the set of nowhere differentiable functions, this gives

$$C([0, 1]) \setminus ND([0, 1]) \subseteq \bigcup_{n \geq 1, m \geq 1} A_{n,m}.$$

Each $A_{n,m}$ is closed. Let $(f_i)_{i \geq 1} \subseteq A_{n,m}$ converge uniformly to f , and let x_i be a witness for f_i . The witnesses all lie in $[0, 1]$, which is closed and bounded, so by the Bolzano–Weierstrass theorem some subsequence converges, say $x_{i_k} \rightarrow x \in [0, 1]$. Fix h with $0 < |h| < 1/m$ and $x+h \in [0, 1]$; for large k the point $x_{i_k} + h$ also lies in $[0, 1]$, after shrinking $|h|$ by an arbitrarily small amount if $x+h$ is an endpoint. Then

$$\left| \frac{f(x+h) - f(x)}{h} \right| = \lim_{k \rightarrow \infty} \left| \frac{f_{i_k}(x_{i_k} + h) - f_{i_k}(x_{i_k})}{h} \right| \leq n,$$

the equality because $f_{i_k} \rightarrow f$ uniformly and $x_{i_k} \rightarrow x$ with f continuous. So x is a witness for f , proving $f \in A_{n,m}$.

Each $A_{n,m}$ contains no open ball. Take any $f \in A_{n,m}$ and any $\varepsilon > 0$; we produce a member of $B(f, \varepsilon)$ that is not in $A_{n,m}$. My first thought was to reach for Fourier series, on the grounds that a truncated series wiggles about the function it approximates.

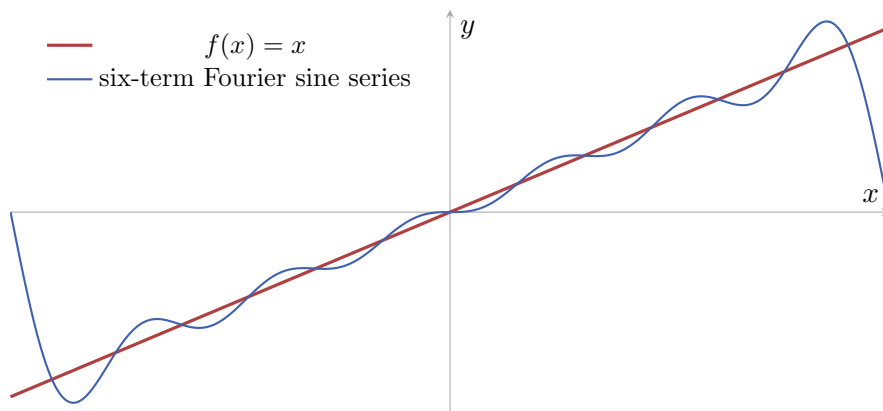


Figure 2: The first six terms of the Fourier sine series of $f(x) = x$. It oscillates about the line, but each partial sum is a finite trigonometric sum, so its derivative is bounded — precisely what we need to avoid.

That does not quite do the job: a partial sum is smooth with bounded derivative, so nothing stops it from lying in $A_{n,m}$ as well. Sawtooth functions do, because their slopes are ours to choose. The piecewise-linear functions are dense in $C([0, 1])$ — f is uniformly continuous on a compact interval, so a fine enough polygonal interpolation stays within ε of it — and a polygonal path can be replaced by a zigzag with the same endpoints, arbitrarily fine teeth, and slopes as steep as we please.

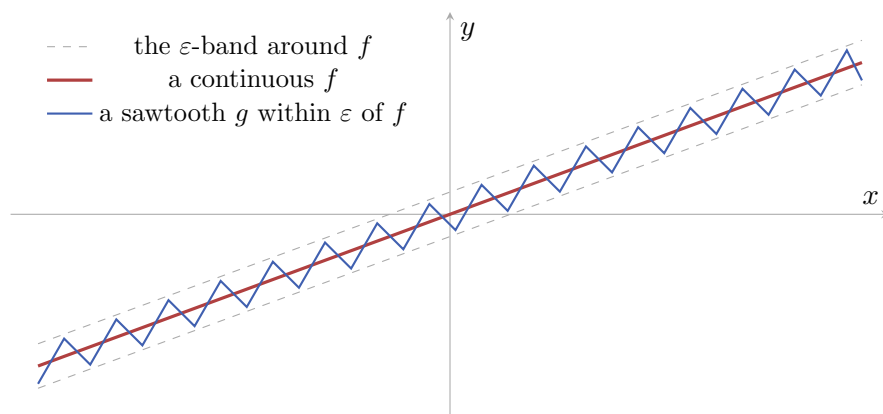


Figure 3: A sawtooth g tracking f to within ε while every one of its segments has slope steeper than n . Narrowing the teeth makes the slopes grow without moving g out of the band.

So choose a sawtooth g with $\|f - g\|_\infty < \varepsilon$, every linear piece of slope greater than n in absolute value, and every piece of width less than $1/m$. Given any $x \in [0, 1]$, take h small enough that x and $x + h$ lie on one linear piece; the difference quotient is then exactly that piece's slope, so it exceeds n while $0 < |h| < 1/m$. No x can witness $g \in A_{n,m}$, so $g \notin A_{n,m}$ although $g \in B(f, \varepsilon)$. Hence $A_{n,m}$

contains no open ball, and being closed it is nowhere dense by Lemma 1.2(b). Detailed proofs of the density claims are in Green's notes [1].

Conclusion. The family $\{A_{n,m}\}$ is indexed by $\mathbb{N}_{\geq 1} \times \mathbb{N}_{\geq 1}$, which is countable, and each member is closed and nowhere dense. By Theorem 2.1 their union contains no open ball. Since $C([0,1]) \setminus ND([0,1])$ is a subset of that union, it contains no open ball either, and Lemma 2.3 turns this into the statement that $ND([0,1])$ is dense in $C([0,1])$. In particular it is nonempty. $\square$

So the nowhere differentiable functions are not merely present, they are the generic case. As is usually the case, Baire's category theorem allows us to prove the existence of some element in a space without giving us any intuition about how to formulate such an element.

3 The Banach–Steinhaus Theorem

Definition 3.0. A subset S of a normed vector space is *uniformly bounded* if and only if there exists a constant $C > 0$ such that $\|x\| < C$ for every $x \in S$.

Definition 3.1. Given a collection of bounded linear operators $(f_\alpha)_{\alpha \in A} \subseteq \mathcal{B}(X, Y)$ for normed vector spaces X, Y and an index set A , we say the collection is *pointwise bounded* at $x \in X$ if and only if there exists $C_x > 0$ such that $\|f_\alpha(x)\| < C_x$ for every $\alpha \in A$.

Theorem 3.2 (Banach–Steinhaus). If a collection of bounded linear operators from a Banach space X to a normed vector space Y is pointwise bounded at every $x \in X$, then the collection is uniformly bounded in the operator space $(\mathcal{B}(X, Y), \|\cdot\|_{\mathcal{B}})$.

Proof. Let $(X, \|\cdot\|_X)$ be a Banach space, $(Y, \|\cdot\|_Y)$ a normed space, and A an index set. Assume $T = (f_\alpha)_{\alpha \in A} \subseteq \mathcal{B}(X, Y)$ is pointwise bounded at every $x \in X$. Partition the domain by how large the operators get: for every $k \in \mathbb{N}$ set

$$E_k := \left\{ x \in X : \sup_{f \in T} \|f(x)\|_Y \leq k \right\} = \bigcap_{f \in T} \{x : \|f(x)\|_Y \leq k\}.$$

Since T is pointwise bounded (Definition 3.1), every $x \in X$ has a bound C_x and so lies in E_k for any $k \geq C_x$. Therefore

$$\bigcup_{k \geq 1} E_k = X.$$

Each set $\{x : \|f(x)\|_Y \leq k\}$ is the preimage of a closed set under a continuous map, hence closed, and an intersection of closed sets is closed, so every E_k is closed. According to Corollary 2.2 of Baire's theorem, the E_k cannot all be nowhere dense. So there exists $N \in \mathbb{N}$ for which E_N is not nowhere dense, and since E_N is closed, Lemma 1.2(b) says it contains an open ball. The closure of that ball remains inside the closed set E_N , so we also obtain a closed ball: there exist $b \in E_N$ and $\varepsilon > 0$ with $\overline{B}(b, \varepsilon) \subseteq E_N$.

For a given $z \in X$ with $z \neq 0$, let $z_\varepsilon = \frac{\varepsilon}{\|z\|_X} z$. This resizing of an arbitrary element z allows us to add it to b and remain in the closed ball, since $\|(b + z_\varepsilon) - b\|_X = \|z_\varepsilon\|_X = \varepsilon$. So $b + z_\varepsilon \in E_N$, which bounds $\sup_{f \in T} \|f(b + z_\varepsilon)\|_Y \leq N$. Since $b \in E_N$ as well, linearity and the triangle inequality give

$$\sup_{f \in T} \|f(z_\varepsilon)\|_Y = \sup_{f \in T} \|f(b + z_\varepsilon) - f(b)\|_Y \leq \sup_{f \in T} \|f(b + z_\varepsilon)\|_Y + \sup_{f \in T} \|f(b)\|_Y \leq 2N.$$

Now that the scaled-down version of z has a concrete bound, scale both sides back up using $z = \frac{\|z\|_X}{\varepsilon} z_\varepsilon$:

$$\sup_{f \in T} \|f(z)\|_Y = \frac{\|z\|_X}{\varepsilon} \sup_{f \in T} \|f(z_\varepsilon)\|_Y \leq \frac{2N\|z\|_X}{\varepsilon}.$$

Dividing by $\|z\|_X$ and taking the supremum over $z \neq 0$, the definition of the operator norm gives

$$\|f\|_{\mathcal{B}} = \sup_{z \neq 0} \frac{\|f(z)\|_Y}{\|z\|_X} \leq \frac{2N}{\varepsilon} \quad \text{for every } f \in T.$$

Since $2N/\varepsilon$ is a constant independent of f , the collection T is uniformly bounded by Definition 3.0, proving the Banach–Steinhaus theorem. $\square$

The shape of the argument is worth keeping. Pointwise boundedness is a countable-cover hypothesis in disguise: it says the space is exhausted by the sublevel sets E_k . Completeness then forces one of them to contain a ball, and linearity broadcasts that single ball’s worth of control across the entire space.

References

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